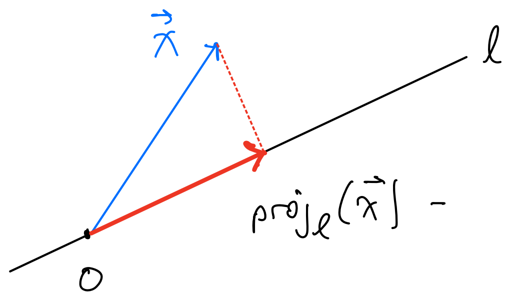
2/6/26

Last time:

Orthogonal Projections in the Plane $\mathbb{R}^2$



$$\boxed{\text{proj}_\ell(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u} \quad \text{where} \quad u = \frac{1}{\|\vec{w}\|}\vec{w}}$$ ← this should be memorized

In fact it's a linear transformation:

say $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

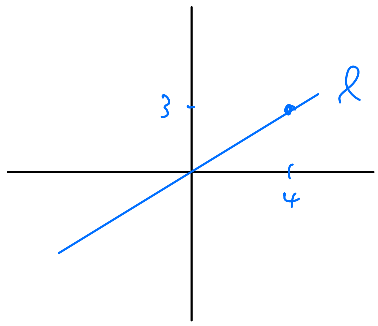
← unit vector

then

$$\text{proj}_\ell(\vec{x}) = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (\star)$$

Ex let ℓ be the line $3x-4y=0$. [The question could also say let ℓ be the line joining 0 and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.]

Find the matrix giving $\text{proj}_\ell(\vec{x})$.



ℓ contains $\vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Note $\|\vec{w}\| = \sqrt{4^2+3^2} = 5$

So $u = \dfrac{\vec{w}}{\|\vec{w}\|} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$ is a unit vector on ℓ.

$\Rightarrow$ $\text{proj}_\ell(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}$ is defined by $(\star)$

$$\text{proj}_\ell\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{(answer)}$$

For the purpose of the next remark, note $\rightarrow$ $\left[\ \frac{4}{25}\begin{bmatrix} 4 \\ 3 \end{bmatrix} \ \Bigg|\ \frac{3}{25}\begin{bmatrix} 4 \\ 3 \end{bmatrix}\ \right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

scalar multiples of the <u>same</u> vector, $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Important observation about projection to a line in $\mathbb{R}^n$:

$$A = \left[\, T(\vec{e}_1) \;\middle|\; \cdots \;\middle|\; T(\vec{e}_n) \,\right]$$

If this represents projection to ℓ (which goes thru the origin) then $T(\vec{e}_1), \ldots, T(\vec{e}_n)$ all lie on ℓ, so the columns of A must be scalar multiples of each other!

The goal of the rest of §2.2 is to similarly analyze similar linear transformations for $\mathbb{R}^2$ <u>and</u> $\mathbb{R}^3$. We'll look at the main ideas.

List of topics:

reflections in $\mathbb{R}^2$ about a line thru the origin

" " $\mathbb{R}^3$ " " plane " " "

" " " " " line " " "

orthog. projections to a plane in $\mathbb{R}^3$ (thru the origin)

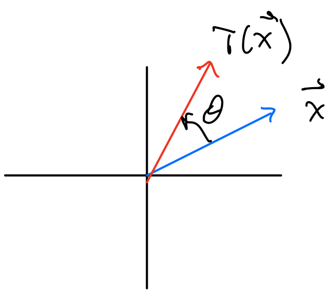
" " " " line in $\mathbb{R}^3$ " " "

rotations in $\mathbb{R}^2$ about the origin

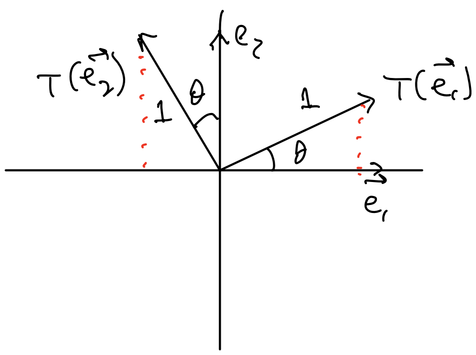
shears in $\mathbb{R}^2$

Plus combinations of these.

Start with <u>rotations</u> in $\mathbb{R}^2$. (Quick)



So focus on $\vec{e}_1$ and $\vec{e}_2$



$$T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

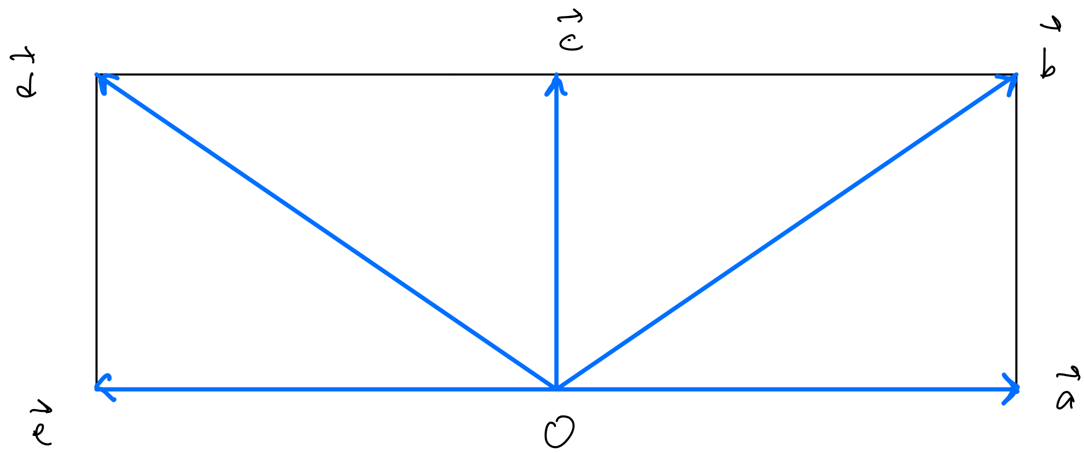
$$T(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

so the matrix for T is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Now come back to the others.

As a preparation, note (assume the rectangles are isomorphic)

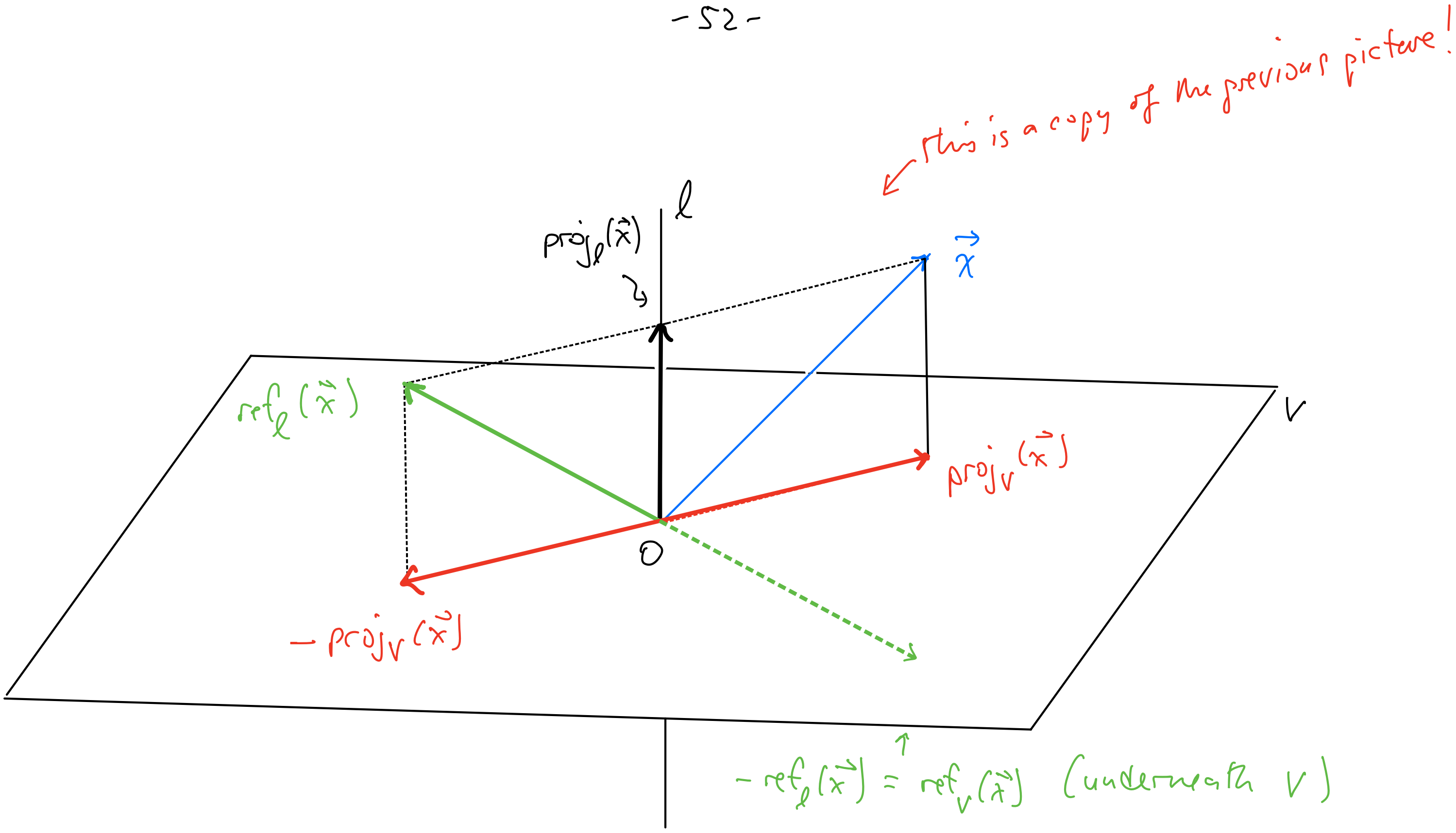


then

- $\vec{a} + \vec{c} = b$
- $\vec{a} + \vec{d} = c$
- $\vec{a} + \vec{e} = \vec{0}$
- $\vec{b} + \vec{d} = 2\vec{c}$ (this takes a moment's thought)

These hold whether the picture is in $\mathbb{R}^2$ or is any plane through the origin in $\mathbb{R}^3$. (or $\mathbb{R}^n$)

Let's look in $\mathbb{R}^3$. Let ℓ be a line thru the origin. Let $V = \ell^\perp$ be the plane thru the origin perp. to ℓ. The following picture defines some of the main constructions



Connections (Don't memorize, but understand.) (Use these for HW.)

The parallelograms are actually rectangles, viewed with perspective.

Let $u = \dfrac{1}{\|\vec{w}\|}\vec{w}$.

Then using the rectangle above we get the following

(0) $\mathrm{proj}_\ell(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}$ where $u = \dfrac{1}{\|\vec{w}\|}\vec{w}$.

(i) $\mathrm{proj}_V(\vec{x}) = \vec{x} - \mathrm{proj}_\ell(\vec{x}) = \vec{x} - (\vec{x}\cdot\vec{u})\vec{u}$

(ii) $\mathrm{ref}_\ell(\vec{x}) + \mathrm{proj}_V(\vec{x}) = \mathrm{proj}_\ell(\vec{x})$

$\Rightarrow \quad \mathrm{ref}_\ell(x) = \mathrm{proj}_\ell(\vec{x}) - \mathrm{proj}_V(\vec{x})$ (but see (iii))

(iii) $\mathrm{ref}_\ell(\vec{x}) + \vec{x} = 2\,\mathrm{proj}_\ell(\vec{x})$

$\Rightarrow \quad \mathrm{ref}_\ell(\vec{x}) = 2\,\mathrm{proj}_\ell(\vec{x}) - \vec{x} = 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}$ (more useful than (ii))

(iv) $\quad \mathrm{ref}_V(\vec{x}) + \mathrm{proj}_\ell(\vec{x}) = \mathrm{proj}_V(\vec{x})$

(v) $\quad \mathrm{ref}_V(\vec{x}) + \mathrm{ref}_\ell(\vec{x}) = 0$

$$\Rightarrow \mathrm{ref}_V(\vec{x}) = \mathrm{proj}_V(\vec{x}) - \mathrm{proj}_\ell(\vec{x}) \qquad \text{(by (iv))}$$

$$= -\mathrm{ref}_\ell(\vec{x}) \qquad \text{(by (v))}$$

$$= \vec{x} - 2(\vec{x}\cdot\vec{u})\vec{u} \qquad \text{(by (iii))}$$